

$$* E = \{1, x, x^2\} \subseteq P_2(\mathbb{R})$$

$$L(a + bx + cx^2) = (a + b + c) + (a + b)x + (a + b)x^2$$

Compute  $\text{Rep}_{E,E}(L)$ .

$$\hookrightarrow \text{Rep}_{B,D}(L)$$

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ U & & U \\ B & & D \end{array}$$

$$\text{Rep}_{B,D}(L) = \left[ [L(b_1)]_D \mid [L(b_2)]_D \mid \dots \mid [L(b_n)]_D \right]$$

$$\text{Want } \text{Rep}_{E,E}(L) = \left[ [L(1)]_E \mid [L(x)]_E \mid [L(x^2)]_E \right]$$

NB:  $L(1) = L(1 \cdot 1 + 0 \cdot x + 0 \cdot x^2) = (1 + 0 + 0) + (1 + 0)x + (1 + 0)x^2 = 1 + x + x^2$

$$* L(a + bx + cx^2) = (a + b + c) + (a + b)x + (a + b)x^2$$

$$L(x) = L(0 \cdot 1 + 1 \cdot x + 0 \cdot x^2) = 1 + x + x^2$$

$$L(x^2) = L(0 \cdot 1 + 0 \cdot x + 1 \cdot x^2) = \underline{1 + 0 \cdot x + 0 \cdot x^2}$$

$$[L(1)]_E = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{array}{l} \text{coeff of } b_1 = 1 \\ \text{coeff of } b_2 = x \\ \text{coeff of } b_3 = x^2 \end{array}$$

$$[L(x)]_E = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$[L(x^2)]_E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Rep}_{E,E}(L) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Remark: this was easy because  $E$  was a very nice basis...

In general, we need to use:



$$\star \text{Rep}_{B,D}(L) = \underbrace{\left[ \text{Rep}_{B',D'}(id) \right]}_a \text{Rep}_{B',D'}(L) \cdot \underbrace{\left[ \text{Rep}_{B,B'}(id) \right]}_c \star$$

$$\begin{array}{ccc} P_2(\mathbb{R})_B & \xrightarrow{\text{Rep}_{B,D}(L)} & P_2(\mathbb{R})_D \\ \downarrow \text{Rep}_{B,E}(id) & & \downarrow \text{Rep}_{D,E}(id) \\ P_2(\mathbb{R})_E & \xrightarrow{\text{Rep}_{E,E}(L)} & P_2(\mathbb{R})_E \end{array}$$

$\text{Rep}_{E,E}(L) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  ← was easy to compute

Orthogonal Complement:  $W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 : \begin{cases} 2a - b + c = 0 \\ a - b - d = 0 \end{cases} \right\}$

$$W = \text{Col}(A) \Rightarrow W^\perp = \text{Null}(A^T)$$

$$\begin{aligned} W &= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{cases} b = 2a + c \\ d = a - b \end{cases} \right\} \\ &= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{cases} b = 2a + c \\ d = -a - c \end{cases} \right\} = \left\{ \begin{bmatrix} a \\ 2a + c \\ c \\ -a - c \end{bmatrix} : a, c \in \mathbb{R} \right\} \end{aligned}$$

$d = a - b = a - (2a + c)$

$$= \left\{ \begin{bmatrix} a \\ 2a \\ 0 \\ -a \end{bmatrix} + \begin{bmatrix} 0 \\ c \\ c \\ -c \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

$$= \text{Col} \left[ \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \right] = \text{Col}(A)$$

$$\therefore W^\perp = \text{null}(A^T)$$

$$= \text{null} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$= \text{null} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -1 \\ a & b & c & d \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{l} a - 2c + d = 0 \\ b + c - d = 0 \end{array} \right\}$$

→ keep going... give a basis ...

$$W = \text{Col}(A) \Rightarrow W^\perp = \text{null}(A^T).$$

Ex: Show  $B = \{1+x, 1+2x+x^2, \underset{3x}{x+2x}\}$  spans  $\underline{\underline{P_2(\mathbb{R})}}$ .

Sol:  $[B]_E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix} = M$

$$\text{RREF}(M) = \text{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} = \text{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} = \text{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

So because  $\text{rank}(M) = \text{rank}(\text{RREF}(M)) = 3$ , and  $\dim(P_2(\mathbb{R})) = 3$ ,  $B$  spans.  $\square$

Ex: Apply Gram-Schmidt process to  $v_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$ ,  $v_2 = \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix}$ .

Sol:  $u_1 = v_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$

$$u_2 = v_2 - \text{proj}_{u_1}(v_2) = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix}}{\begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} - \frac{-15 - 9 - 18 - 3}{9 + 1 + 4 + 1} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} + \frac{3(5+3+6+1)}{15} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 9 \\ -9 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$$

$\therefore u_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$  and  $u_2 = \begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$  span same space as  $v_1, v_2$

and they are orthogonal "

□

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}$$

$\lambda = 2$

$\lambda = 1$

$\lambda = -5$

$\lambda = \pi$

to make orthonormal:

All step:

Normalize each vector  $\rightarrow$

$u_1 \rightarrow w_1 = \frac{1}{|u_1|} u_1$ ,  $u_2 \rightarrow w_2 = \frac{1}{|u_2|} u_2$



In the previous example:  $u_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}$ ,  $u_2 = \begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$

$$|u_1| = \sqrt{3^2 + 1^2 + 2^2 + 1^2} = \sqrt{15}, \quad |u_2| = \sqrt{4^2 + 6^2 + 3^2} = \sqrt{5^2 + 6^2} = \sqrt{61}$$

$$\therefore w_1 = \frac{1}{|u_1|} u_1 = \frac{1}{\sqrt{15}} \begin{pmatrix} 3 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \quad w_2 = \frac{1}{|u_2|} u_2 = \frac{1}{\sqrt{61}} \begin{pmatrix} 4 \\ 6 \\ -3 \\ 0 \end{pmatrix}$$

is orthonormal collection spanning same space as  $v_1, v_2$ . □

$$\begin{array}{ccc} V_B & \xrightarrow{\text{Rep}_{B,D}(L)} & W_D \\ \sim \downarrow & & \downarrow \sim \\ \mathbb{R}^{\dim(V)} & \xrightarrow{\quad} & \mathbb{R}^{\dim(W)} \\ & M = \text{Rep}_{B,D}(L) & \end{array}$$

$L: V \rightarrow W$   
isomorphism, iff  
 $L^{-1}$  exists.

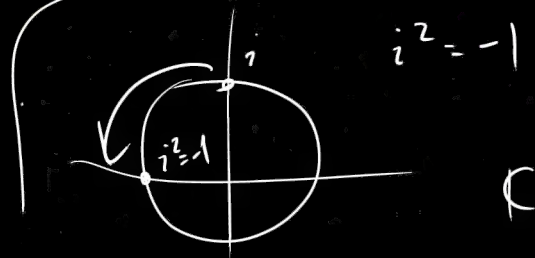
\*  $L(B) = \underline{X}$  ← basis of  $W$  iff  $L$  is an iso

\*  $L^{-1}: W \rightarrow V$  via  $\underline{L^{-1}(L(b)) = b}$ .

extend linearly (using lin. ext prop. from before Exan 2).

Ex:  $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  has e-values  $\lambda = \pm i$  (check).

is NOT diagonalizable over  $\mathbb{R}$ ,  
but IS diagonalizable over  $\mathbb{C}$



$$W = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a + b = 0 \right\}$$